

# Equitable chromatic threshold of complete multipartite graphs

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## Abstract

A proper vertex coloring of a graph is equitable if the sizes of color classes differ by at most one. The equitable chromatic number of a graph  $G$ , denoted by  $\chi_=(G)$ , is the minimum  $k$  such that  $G$  is equitably  $k$ -colorable. The equitable chromatic threshold of a graph  $G$ , denoted by  $\chi_=(G)$ , is the minimum  $t$  such that  $G$  is equitably  $k$ -colorable for  $k \geq t$ . We develop a formula and a linear-time algorithm which compute the equitable chromatic threshold of an arbitrary complete multipartite graph.

*Keywords:* equitable coloring, equitable chromatic threshold, complete multipartite graphs

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## 1. Introduction

All graphs considered in this paper are finite, undirected and without loops or multiple edges. For a positive integer  $k$ , let  $[k] = \{1, 2, \dots, k\}$ . A proper  $k$ -coloring of a graph  $G$  is a mapping  $f : V(G) \rightarrow [k]$  such that  $f(x) \neq f(y)$  whenever  $xy \in E(G)$ . We call the set  $f^{-1}(i) = \{x \in V(G) : f(x) = i\}$  a color class for each  $i \in [k]$ . A graph is  $k$ -colorable if it has a  $k$ -coloring. The chromatic number of  $G$ , denoted by  $\chi(G)$ , is equal to  $\min\{k : G \text{ is } k\text{-colorable}\}$ . An equitable  $k$ -coloring of  $G$  is a  $k$ -coloring for which any two color classes differ in size by at most one, or equivalently, each color class is of size  $\lfloor |V(G)|/k \rfloor$  or  $\lceil |V(G)|/k \rceil$ . If  $G$  has  $n$  vertices, we write  $n = kq + r$

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with  $0 \leq r < k$ , then we can rewrite  $n = (k - r)q + r(q + 1)$ , or equivalently, exactly  $r$  (respectively,  $k - r$ ) color classes have size  $q + 1$  (respectively,  $q$ ). The equitable chromatic number of  $G$ , denoted by  $\chi_=(G)$ , is equal to  $\min\{k : G \text{ is equitably } k\text{-colorable}\}$ , and the equitable chromatic threshold of a graph  $G$ , denoted by  $\chi_*(G)$ , is equal to  $\min\{t : G \text{ is equitably } k\text{-colorable for } k \geq t\}$ .

The concept of equitable colorability was first introduced by Meyer [21]. The definitive survey of the subject is by Lih [19]. Many application such as scheduling and constructing timetables, please see [1, 9, 10, 12, 23, 25, 26].

In 1964, Erdős [6] conjectured that any graph  $G$  with maximum degree  $\Delta(G) \leq k$  has an equitable  $(k + 1)$ -coloring, or equivalently,  $\chi_*(G) \leq \Delta(G) + 1$ . This conjecture was proved in 1970 by Hajnal and Szemerédi [8] with a long and complicated proof, a polynomial algorithm for such a coloring was found by Mydlarz and Szemerédi [22]. Kierstead and Kostochka [11] gave a short proof of the theorem, and presented another polynomial algorithm for such a coloring. Brooks' type results are conjectured: Equitable Coloring Conjecture [21]  $\chi_=(G) \leq \Delta(G)$ , and Equitable  $\Delta$ -Coloring Conjecture [4]  $\chi_*(G) \leq \Delta(G)$  for  $G \notin \{K_n, C_{2n+1}, K_{2n+1, 2n+1}\}$ . Exact values of equitable chromatic numbers of trees [3] and complete multipartite graphs [2], [18] were determined. Our article determines the exact value of equitable chromatic threshold of complete multipartite graphs.

The formula which is different from ours was established independently in a manuscript by Chen and Wu, and was reported in [19]. However, Chen and Wu never published their proof. To our knowledge, this article contains the only published proof.

## 2. The results

Before stating our main result, we need several preliminary results on integer partitions. Recall that a partition of an integer  $n$  is a sum of the form  $n = m_1 + m_2 + \cdots + m_k$ , where  $0 \leq m_i \leq n$  for each  $0 \leq i \leq k$ . We call such a partition a  $q$ -partition if each  $m_i$  is in the set  $\{q, q + 1\}$ . A  $q$ -partition of  $n$  is typically denoted as  $n = aq + b(q + 1)$ , where  $n$  is the sum of  $a$   $q$ 's and  $b$   $q + 1$ 's. A  $q$ -partition of  $n$  is called a minimal  $q$ -partition if the number of its addends,  $a + b$ , is as small as possible. A  $q$ -partition of  $n$  is called a maximal  $q$ -partition if the number of its addends,  $a + b$ , is as large as possible. For example,  $2 + 2 + 2 + 2$  is a maximal 2-partition of 8, and  $2 + 3 + 3$  is a minimal 2-partition of 8. If  $q|n$ , or equivalently,  $n = kq$ , with

$k \geq 1$ , thus we write  $n = 0(q-1) + kq$  (respectively,  $n = kq + 0(q+1)$ ), then there are both  $(q-1)$ -partition and  $q$ -partition of  $n$ . For example, since  $2 \mid 8$ , we write  $8 = 0 \times 1 + 4 \times 2$  (respectively,  $8 = 4 \times 2 + 0 \times 3$ ), then there are both 1-partition and 2-partition of 8.

Our first lemma is from [2], which study the condition of which a  $q$ -partition of  $n$  exists. For the sake of completeness, here we restate their proof. In what follows, all variables are nonnegative integers.

**Lemma 2.1.** [2] *If  $0 < q \leq n$ , and  $n = kq + r$  with  $0 \leq r < q$ , then there is a  $q$ -partition of  $n$  if and only if  $r \leq k$ .*

*Proof.* If  $r \leq k$ , then  $n = (k-r)q + r(q+1)$  is a  $q$ -partition of  $n$ . Conversely, given a  $q$ -partition  $n = aq + b(q+1)$  of  $n$ , we have  $n = (a+b)q + b$ , so  $(a+b) \leq k$  and  $r \leq b$ . Consequently,  $r \leq b \leq (a+b) \leq k$ .  $\square$

**Corollary 2.1.** *There is no  $q$ -partition of  $n$  if and only if  $n/(q+1) > \lfloor n/q \rfloor$ .*

*Proof.* Using the division algorithm, write  $n = kq + r$ , with  $0 \leq r < q$ . Then  $k = \lfloor n/q \rfloor$ , and  $r = n - \lfloor n/q \rfloor q$ . Lemma 2.1 implies that there is no  $q$ -partition of  $n$  if and only if  $r > k$ , hence  $n - \lfloor n/q \rfloor q > \lfloor n/q \rfloor$ , we can rewrite  $n > \lfloor n/q \rfloor (q+1)$ . The Corollary 2.1 follows immediately.  $\square$

The next two lemmas give conditions under which a  $q$ -partition of  $n$  is maximal (respectively, minimal).

**Lemma 2.2.** *A  $q$ -partition  $n = aq + b(q+1)$  of  $n$  is maximal if and only if  $b < q$ . Moreover a maximal  $q$ -partition is unique.*

*Proof.* Regard  $a$  and  $b$  as variables, and  $q$  as fixed. Solving the linear relation  $n = aq + b(q+1)$  for  $a$  yields  $a+b = (n-b)/q$ . Thus  $a+b$  is a strictly decreasing function of  $b$ , and moreover  $a+b$  decreases as  $b$  increases. Therefore, the  $q$ -partition  $n = aq + b(q+1)$  will be maximal exactly when  $b$  is the smallest non-negative integer for which  $(n-b)/q$  is an integer. Once  $b$  is fixed,  $a$  is determined by the equation  $n = aq + b(q+1)$ . Uniqueness of maximal  $q$ -partition follows.

Now suppose  $n = aq + b(q+1)$  is a  $q$ -partition, and  $b < q$ . By what was said in the previous paragraph,  $m = (n-b)/q$  is an integer. If the partition is not maximal, then there are integers  $b'$  and  $m'$ , with  $b > b' \geq 0$  and  $m' > m > 0$ , for which  $m' = (n-b')/q$ . Subtracting  $n = m'q + b'$  from  $n = mq + b$  gives  $b - b' = (m' - m)q$ , so  $b > (b - b') \geq q$ .

Conversely, if  $n = aq + b(q + 1)$  is a maximal  $q$ -partition of  $n$ , it is impossible for  $b \geq q$ , for otherwise  $n = (a + q + 1)q + (b - q)(q + 1)$  is a  $q$ -partition of  $n$  with  $a + q + 1 + b - q = a + b + 1$  addends, contradicting maximality. Thus,  $b < q$ .  $\square$

**Lemma 2.3.** [2] *A  $q$ -partition  $n = aq + b(q + 1)$  of  $n$  is minimal if and only if  $a < q + 1$ . Moreover a minimal  $q$ -partition is unique.*

Now it is possible to describe exactly the number of addends in a maximal (respectively, minimal)  $q$ -partition.

**Lemma 2.4.** *If  $n = aq + b(q + 1)$  is a minimal  $q$ -partition, then  $a + b = \lceil n/(q + 1) \rceil$ . If  $n = a'q + b'(q + 1)$  is a maximal  $q$ -partition, then  $a' + b' = \lfloor n/q \rfloor$ . Moreover, when  $\lceil n/(q + 1) \rceil = \lfloor n/q \rfloor$ , there is only one  $q$ -partition of  $n$ .*

*Proof.* If  $n = aq + b(q + 1)$  is a minimal  $q$ -partition, then  $a + b = (n + a)/(q + 1)$ , with  $a < q + 1$  by Lemma 2.3, and so  $a + b = \lceil n/(q + 1) \rceil$ . If  $n = a'q + b'(q + 1)$  is a maximal  $q$ -partition, then  $a' + b' = (n - b')/q$ , with  $b' < q$  by Lemma 2.2, and so  $a' + b' = \lfloor n/q \rfloor$ . Now, if  $\lceil n/(q + 1) \rceil = \lfloor n/q \rfloor$ , then  $a + b = a' + b'$ . From Lemma 2.3 and Lemma 2.2, we know that the minimal (respectively, maximal)  $q$ -partition is unique. Consequently, if  $\lceil n/(q + 1) \rceil = \lfloor n/q \rfloor$ , then there is only one  $q$ -partition of  $n$ .  $\square$

**Lemma 2.5.** *Let  $n = aq + b(q + 1)$  be the maximal  $q$ -partition, and  $n = a'(q - 1) + b'q$  be the minimal  $(q - 1)$ -partition. If  $q|n$  then  $a + b = a' + b'$ , otherwise,  $a + b + 1 = a' + b'$ .*

*Proof.* By Lemma 2.4, if  $n = aq + b(q + 1)$  is the maximal  $q$ -partition, then  $a + b = \lfloor n/q \rfloor$ . If  $n = a'(q - 1) + b'q$  is the minimal  $(q - 1)$ -partition, Lemma 2.4 implies that  $a' + b' = \lceil n/(q - 1 + 1) \rceil = \lceil n/q \rceil$ . Consequently, if  $q|n$  then  $a + b = \lfloor n/q \rfloor = \lceil n/q \rceil = a' + b'$ , otherwise,  $a' + b' = \lceil n/q \rceil = \lfloor n/q \rfloor + 1 = a + b + 1$ .  $\square$

If  $n_1 = a_1q + b_1(q + 1)$ , and  $n_2 = a_2q + b_2(q + 1)$  are maximal  $q$ -partition of  $n_1$  and  $n_2$ , respectively. If  $n_1 = a'_1(q - 1) + b'_1q$ , and  $n_2 = a'_2(q - 1) + b'_2q$  are minimal  $(q - 1)$ -partition of  $n_1$  and  $n_2$ , respectively. Lemma 2.5 implies that

$$a'_1 + b'_1 + a'_2 + b'_2 = \begin{cases} a_1 + b_1 + a_2 + b_2 & , \quad q|n_1 \text{ and } q|n_2 \\ a_1 + b_1 + a_2 + b_2 + 2, & q \nmid n_1 \text{ and } q \nmid n_2 \\ a_1 + b_1 + a_2 + b_2 + 1, & (q \nmid n_1 \text{ and } q|n_2) \text{ or } (q|n_1 \text{ and } q \nmid n_2). \end{cases}$$

These results now combine to give a construction of a minimal equitable coloring of  $K_{n_1, n_2, \dots, n_l}$  and a method to change the color classes step by step, so that we can increase the equitable colors one by one. In words, we must give the computation of the minimum  $t$ , when  $K_{n_1, n_2, \dots, n_l}$  can be equitably  $k$ -colorable for  $k \geq t$ .

Denote the partite sets of the graph  $K_{n_1, n_2, \dots, n_l}$  as  $N_1, N_2, \dots, N_l$ , with  $|N_i| = n_i$ . Any given color class of an equitable coloring must lie entirely in some  $N_i$ , for otherwise two of its vertices are nonadjacent. Thus, any equitable coloring partitions each  $N_i$  into color classes  $V_{i_1}, V_{i_2}, \dots, V_{i_{v_i}}$ , no two of which differ in size by more than one. If the sizes of the color classes are in the set  $\{q, q+1\}$ , then these sizes induce  $q$ -partitions of each  $n_i$ . Conversely, given a number  $q$ , and  $q$ -partitions  $n_i = a_i q + b_i(q+1)$ , of each  $n_i$ , there is an equitable coloring of  $K_{n_1, n_2, \dots, n_l}$  with color sizes  $q$  and  $q+1$ ; just partition each  $N_i$  into  $a_i$  sets of size  $q$ , and  $b_i$  sets of  $q+1$ . It follows, then, that finding an equitable coloring of  $K_{n_1, n_2, \dots, n_l}$  amounts to finding a number  $q$ , and simultaneous  $q$ -partitions of each of numbers  $n_i$ . By Corollary 2.1, a necessary condition for  $q$  is that  $n_i/(q+1) \leq \lfloor n_i/q \rfloor$  for all  $1 \leq i \leq l$ . If we want increase colors one by one,  $q$  must be chosen with the additional property that the total number of color classes is as small as possible. By Lemmas 2.1, 2.4 and 2.5, it suffices to choose the minimum  $q$  for which there is  $i$  such that  $n_i/(q+1) > \lfloor n_i/q \rfloor$  or there are  $n_i$  and  $n_j$ , such that  $q$  divides neither  $n_i$  nor  $n_j$ . Equivalently, it suffices to choose the maximum  $q-1$  for which there is  $i$  such that  $n_i/q \leq \lfloor n_i/(q-1) \rfloor$ , and  $(q-1)|n_j$ , for  $j \neq i$ . Moreover we can partition each  $n_i$  into  $a_i = q \lfloor n_i/q \rfloor - n_i$  of sizes  $q-1$ , and  $b_i = n_i - \lfloor n_i/q \rfloor (q-1)$  of sizes  $q$ .

**Theorem 2.1.**  $\chi_{=}^*(K_{n_1, n_2, \dots, n_l}) = \sum_{i=1}^l \lceil n_i/h \rceil$ , where  $h = \min\{q : \text{there is } i \text{ such that } n_i/(q+1) > \lfloor n_i/q \rfloor \text{ or there are } n_i \text{ and } n_j, i \neq j, \text{ such that } q \text{ divides neither } n_i \text{ nor } n_j\}$ .

*Proof.* We prove that  $K_{n_1, n_2, \dots, n_l}$  is equitably  $k$ -colorable for any  $k \geq \sum_{i=1}^l \lceil n_i/h \rceil$  by induction on  $k$ .

First, we prove that  $K_{n_1, n_2, \dots, n_l}$  is equitably  $\sum_{i=1}^l \lceil n_i/h \rceil$ -colorable. Set  $h' = h - 1$ , by the definition of  $h$ ,  $n_i/(h'+1) \leq \lfloor n_i/h' \rfloor$ , for  $1 \leq i \leq l$ . Corollary 2.1 implies that each  $n_i$  has an  $h'$ -partition. Let  $n_i = a_i h' + b_i(h'+1)$  be the minimal  $h'$ -partition of each  $n_i$ . By Lemma 2.4,  $a_i + b_i = \lceil n_i/(h'+1) \rceil = \lceil n_i/h \rceil$ , and hence we get an equitable  $\sum_{i=1}^l \lceil n_i/h \rceil$ -coloring of  $K_{n_1, n_2, \dots, n_l}$ . It is straightforward to check that  $K_{n_1, n_2, \dots, n_l}$  is equitably  $\sum_{i=1}^l \lceil n_i/h \rceil$ -colorable.

Now, we assume that  $K_{n_1, n_2, \dots, n_l}$  is equitably  $k$ -colorable for some  $k \geq \sum_{i=1}^l \lceil n_i/h \rceil$ . It suffices to prove  $K_{n_1, n_2, \dots, n_l}$  is equitably  $(k+1)$ -colorable.

By the assumption, each  $n_i$  has a  $q$ -partition  $n_i = a_i q + b_i(q+1)$  such that  $\sum_{i=1}^l (a_i + b_i) = k$ .

**Claim 1**  $0 \leq q \leq h-1 < h$ .

Suppose to the contrary that  $q \geq h$ . By Lemma 2.4,  $a_i + b_i \leq \lfloor n_i/q \rfloor$ , and hence  $k = \sum_{i=1}^l (a_i + b_i) \leq \sum_{i=1}^l \lfloor n_i/q \rfloor \leq \sum_{i=1}^l n_i/q \leq \sum_{i=1}^l n_i/h$ . By the definition of  $h$ , there are  $n_i$  and  $n_j$ ,  $i \neq j$ , such that  $h$  divides neither  $n_i$  nor  $n_j$ , or there is some  $n_i$  such that  $n_i/(h+1) > \lfloor n_i/h \rfloor$ . Either case implies that  $h \nmid n_i$  for some  $n_i$ . Hence  $k \leq \sum_{i=1}^l n_i/h < \sum_{i=1}^l \lceil n_i/h \rceil$ . This is a contradiction to  $k \geq \sum_{i=1}^l \lceil n_i/h \rceil$ . The claim follows.

To prove  $K_{n_1, n_2, \dots, n_l}$  is equitably  $(k+1)$ -colorable, we consider two cases.

**Case 1:** There is some  $n_i$  such that whose  $q$ -partition  $n_i = a_i q + b_i(q+1)$  is not maximal. By Lemma 2.2,  $b_i \geq q$ , so we can rewrite  $n_i = (a_i + q + 1)q + (b_i - q)(q+1)$ . Thus there is a  $q$ -partition of  $n_i$  with  $a_i + q + 1 + b_i - q = a_i + 1 + b_i$  addends. Hence, we get an equitable  $(k+1)$ -coloring of  $K_{n_1, n_2, \dots, n_l}$ .

**Case 2:** Each  $q$ -partition  $n_i = a_i q + b_i(q+1)$  is maximal. By Claim 1,  $0 \leq q \leq h-1 < h$ , the definition of  $h$  implies that  $q$  divides  $n_i$  for all  $i$  with at most one exception.

**Subcase 2.1:** There is no  $i$  such that  $q \nmid n_i$ , in other words,  $q \mid n_i$  for all  $i$ . By Lemma 2.5, each maximal  $q$ -partition is the minimal  $(q-1)$ -partition of  $n_i$ . Since  $0 \leq q-1 \leq h-2 < h$ , it implies that  $q-1$  divides  $n_i$  for all  $i$  with at most one exception. Consequently, there is some  $n_j$  such that  $(q-1) \mid n_j$  and  $q \nmid n_j$ , and the number of addends of minimal (respectively, maximal)  $(q-1)$ -partition is equal to  $\lceil n_j/q \rceil = n_j/q$  (respectively,  $\lfloor n_j/(q-1) \rfloor = n_j/(q-1)$ ) by Lemma 2.4. Since  $n_j/(q-1) > n_j/q$ , the minimal  $(q-1)$ -partition is not the maximal  $(q-1)$ -partition of  $n_j$ . Thus, the minimal  $(q-1)$ -partition of  $n_j$  is just not maximal. So it turn into case 1. So we can obtain an equitable  $(k+1)$ -coloring of  $K_{n_1, n_2, \dots, n_l}$ .

**Subcase 2.2:** There is exactly an  $i$  such that  $q \nmid n_i$ , and at the same time,  $q \mid n_j$  for  $j \neq i$ , with  $1 \leq j \leq l$ . By Lemma 2.5, each maximal  $q$ -partition of  $n_j$  is the minimal  $(q-1)$ -partition of  $n_j$ . Since  $q \nmid n_i$ , and  $q < h$ , by the definition of  $h$ ,  $n_i/q \leq \lfloor n_i/(q-1) \rfloor$ . Corollary 2.1 implies that  $n_i$  has a  $(q-1)$ -partition. Let the partition  $n_i = a_i q + b_i(q+1)$  (respectively, the partition  $n_i = a'_i(q-1) + b'_i q$ ) be the maximal  $q$ -partition (respectively, the minimal  $(q-1)$ -partition) of  $n_i$ , the number of addends  $a'_i + b'_i$  is equal to  $a_i + b_i + 1$  by Lemma 2.4. So we obtain an equitable  $k+1$ -coloring of

$K_{n_1, n_2, \dots, n_l}$ .

In a word, we have proved that  $\chi_=(K_{n_1, n_2, \dots, n_l}) \leq \sum_{i=1}^l \lceil n_i/h \rceil$ .

Next we prove that  $K_{n_1, n_2, \dots, n_l}$  is not equitably  $(\sum_{i=1}^l \lceil n_i/h \rceil - 1)$ -colorable.

Suppose to the contrary that  $K_{n_1, n_2, \dots, n_l}$  is equitably  $(\sum_{i=1}^l \lceil n_i/h \rceil - 1)$ -colorable. Then, each  $n_i$  has a  $q$ -partition  $n_i = a_i q + b_i(q + 1)$  such that  $k = \sum_{i=1}^l (a_i + b_i) = \sum_{i=1}^l \lceil n_i/h \rceil - 1$ .

**Claim 2**  $q = h$

First, we prove that  $q \geq h$ . Suppose to the contrary that  $q \leq h - 1 < h$ . By Lemma 2.4,  $(a_i + b_i) \geq \lceil n_i/(q + 1) \rceil$ , thus  $\sum_{i=1}^l (a_i + b_i) \geq \sum_{i=1}^l \lceil n_i/(q + 1) \rceil \geq \sum_{i=1}^l \lceil n_i/h \rceil$ . This is a contradiction to  $k = \sum_{i=1}^l \lceil n_i/h \rceil - 1$ . Second, we prove that  $q \leq h$ . Suppose to the contrary that  $q > h$ . Lemma 2.4 implies that  $(a_i + b_i) \leq \lfloor n_i/q \rfloor < \lfloor n_i/h \rfloor$ . By the definition of  $h$ , there is some  $n_i$  such that  $n_i \nmid h$ , clearly,  $\lceil n_i/h \rceil - 1 = \lfloor n_i/h \rfloor$ . Thus,  $k < \sum_{i=1}^l \lceil n_i/h \rceil - 1$ . This is a contradiction to  $k = \sum_{i=1}^l \lceil n_i/h \rceil - 1$ . The claim follows.

Now, we consider two cases of  $h$ .

**case 1:**  $h = \min\{q : \text{there is } i \text{ such that } n_i/(q + 1) > \lfloor n_i/q \rfloor\}$ . By Corollary 2.1, there is no  $h$ -partition of  $n_i$ . It is contrary to that each  $n_i$  is partitioned into sets of the sizes  $h$  or  $h + 1$ .

**case 2:**  $h = \min\{q : \text{there are } n_i \text{ and } n_j, i \neq j, \text{ such that } q \text{ divides neither } n_i \text{ nor } n_j\}$ . Let  $n_i = a'_i(h - 1) + b'_i h$ ,  $n_j = a'_j(h - 1) + b'_j h$  be the minimal  $(h - 1)$ -partition of  $n_i$  and  $n_j$ , respectively. Let  $n_i = a_i h + b_i(h + 1)$ ,  $n_j = a_j h + b_j(h + 1)$  be the maximal  $h$ -partition of  $n_i$  and  $n_j$ , respectively. Lemma 2.5 implies that  $a_i + b_i + a_j + b_j = a'_i + b'_i + a'_j + b'_j - 2$ . And hence,  $\sum_{i=1}^l (a_i + b_i) \leq \sum_{i=1}^l \lceil n_i/h \rceil - 2$ . Consequently, we can not obtain an equitable  $(\sum_{i=1}^l \lceil n_i/h \rceil - 1)$ -coloring of  $K_{n_1, n_2, \dots, n_l}$ .

Therefore,  $\chi_=(K_{n_1, n_2, \dots, n_l}) \geq \sum_{i=1}^l \lceil n_i/h \rceil$ , and so  $\chi_=(K_{n_1, n_2, \dots, n_l}) = \sum_{i=1}^l \lceil n_i/h \rceil$ .  $\square$

Theorem 2.1 leads immediately to an algorithm which finds the minimal equitable coloring of  $K_{n_1, n_2, \dots, n_l}$  such that we can increase the colors one by one, through we adjust the partition of  $n_i$  step by step.

### Equitable Chromatic threshold algorithm

Let  $K_{n_1, n_2, \dots, n_l}$  be a complete multipartite graph, where the partite sets of the graph  $K_{n_1, n_2, \dots, n_l}$  are denoted as  $N_1, N_2, \dots, N_l$ , with  $|N_i| = n_i$ . Let  $s^* = \min\{s_i^*, \text{ where } s_i^* \text{ is the minimum positive integer such that } s_i^* \nmid n_i\}$ .

(0) Set  $h = s^*$ .

- (1) If there are  $n_i$  and  $n_j$ , such that  $h$  divides neither  $n_i$  nor  $n_j$ , with  $i \neq j$ , stop. Otherwise, go to (2).
- (2) There is  $i$  such that  $h \nmid n_i$ , and  $h \mid n_j$  with  $i \neq j$ . If  $n_i/(h+1) > \lfloor n_i/h \rfloor$ , stop. Otherwise, go to (3).
- (3) Let  $h = h + 1$ , go to (1).

The equitable Chromatic threshold of  $K_{n_1, n_2, \dots, n_l}$  is  $\sum_{i=1}^l \lceil n_i/h \rceil$ . Notice that the complexity of the algorithm is linear in  $|V(K_{n_1, n_2, \dots, n_l})|$ .

According Theorem 2.1, we have the following corollary which is a W.-H. Lin's[31] result.

**Corollary 2.2.** [31] *If integers  $n \geq 1$  and  $r \geq 2$ , then  $\chi^*(K_{\underbrace{n, n, \dots, n}_r}) = r \lceil n/s^* \rceil$ , where  $s^*$  is the minimum positive integer such that  $s^* \nmid n$ .*

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